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The diagonal subalgebra of a blow-up algebra

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Abstract

Given a bigraded k-algebra $S = \bigoplus_{(u,v)} S_{(u,v)}$, $(u,v) \in \mathbb{N} \times \mathbb{N}$, ($k$ a field), one attaches to it the so-called diagonal subalgebra $S_\Delta = \bigoplus_{(u,u)} S_{(u,u)}$. This notion generalizes the concept of Segre product of graded algebras. The classical situation has $S = k[S_{(1,0)}, S_{(0,1)}]$, whereby taking generators of $S_{(1,0)}$ and $S_{(0,1)}$ yields a closed embedding $\operatorname{Proj}(S) \hookrightarrow \mathbb{P}_k^{n-1} \times \mathbb{P}_k^{r-1}$, for suitable n, r; the resulting generators of $S_{(1,1)}$ make S_Δ isomorphic to the homogeneous coordinate ring of the image of $\operatorname{Proj}(S)$ under the Segre map $\mathbb{P}_k^{n-1} \times \mathbb{P}_k^{r-1} \to \mathbb{P}_k^{nr-1}$.

The main results of this paper deal with the situation where S is the Rees algebra of a homogeneous ideal generated by polynomials in a fixed degree. In this framework, S_Δ is a standard graded algebra which, in some case, can be seen as the homogeneous coordinate ring of certain rational varieties embedded in projective space. This includes some examples of rational surfaces in $\mathbb{P}_k^5$ and toric varieties in $\mathbb{P}_k^n$. The main concern is then with the normality and the Cohen–Macaulayness of S_Δ. One can describe the integral closure of S_Δ explicitly in terms of the given ideal and show that normality carries from S to S_Δ. In contrast to normality, Cohen–Macaulayness fails to behave similarly, even in the case of the Segre product of Cohen–Macaulay graded algebras. The problem is rather puzzling, but one is able to treat a few interesting classes of ideals under which the corresponding Rees algebras yield Cohen–Macaulay diagonal subalgebras. These classes include complete intersections and determinantal ideals generated by the maximal minors of a generic matrix. © 1998 Elsevier Science B.V.

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1. Introduction

Let k be an algebraically closed field and let $\mathbb{P}^s = \mathbb{P}^s_k$ denote projective s-space over k. Given subvarieties $V \subset \mathbb{P}^{n-1}$ and $W \subset \mathbb{P}^{r-1}$, one can look at the image of $V \times W$ under the classical Segre embedding $\mathbb{P}^{n-1} \times \mathbb{P}^{r-1} \hookrightarrow \mathbb{P}^{nr-1}$, the so-called *Segre product* of V and W. Much has been said about the finer arithmetical properties of the homogeneous coordinate ring of the Segre product (cf. [7, 12, 23]).

The product $V \times W$ is only a special case of a subvariety of $\mathbb{P}^{n-1} \times \mathbb{P}^{r-1}$ which is defined by a bihomogeneous ideal J in the natural bigradation of $k[\mathbf{X},\mathbf{T}]$, where $\mathbf{X} = \{X_1,\dots,X_n\}$, $\mathbf{T} = \{T_1,\dots,T_r\}$. These varieties are classically known as *correspondences* and their importance in Intersection Theory cannot be exagerated. To our knowledge, however, a systematic study of the finer arithmetical properties of the homogeneous coordinate ring of the image of such a subvariety in $\mathbb{P}^{nr-1}$ has never been fully taken up.

If $S = k[\mathbf{X},\mathbf{T}]/J$ is the bihomogeneous coordinate ring of a correspondence, S_Δ denotes the corresponding diagonal subalgebra. For our purpose, k may well be an arbitrary field.

In the first section one collects general facts about the diagonal subalgebra S_Δ of a bigraded k-algebra S. Namely, one compares the two algebras in terms of presentation, dimension and multiplicity.

The main feature is about the Cohen–Macaulayness of S_Δ. In the case of Segre products this is a classic by Chow [7], so one would expect some interesting obstructions. The result of Chow's was recast in a different form and translated into modern numerical conditions by Stückrad–Vogel [23] and Goto–Watanabe [12]. In this work, one uses certain filtrations on $k[\mathbf{X},\mathbf{T}]$ to reduce the problem to a special situation where the diagonal subalgebra becomes actually a Segre product and then uses the criterion for Cohen–Macaulayness in this case. The first main point is the description of the initial ideal of J under the aforementioned filtration. Such a description has been recently obtained in [14], in the case of the Rees algebra of an ideal generated by a d-sequence, and subsequently, in [20], for the class of ideals possessing a set of generators forming a *quadratic sequence* and in [24], for ideals of the principal class.

Next one is able to express the integral closure of S_Δ in terms of the integral closure of S, thereby showing that normality carries over from S to S_Δ. This result also follows from the existence of a Reynolds operator from S to S_Δ. One notices here a certain "cylinder phenomenon" related to both the normality and the Cohen–Macaulayness. Actually, this is at the root of the original considerations of Chow, in the case of Segre products, though he did not explicitly put it this way.

Among the important correspondences in algebraic geometry, blowing-up varieties dominate. The subsequent section will be focused on the standard bigraded Rees algebra $\mathcal{R}(I)$ of a homogeneous ideal $I \subset k[\mathbf{X}] = k[X_1,\dots,X_n]$ generated by polynomials of the same degree d. In this case, the diagonal subalgebra S_Δ of $\mathcal{R}(I)$ can be identified with the homogeneous coordinate ring $k[\mathbf{X}I_d]$ of the special fiber of the blow-up (Rees algebra) of the non-saturated ideal $(\mathbf{X})I$. A very first non-trivial example of such a diagonal subalgebra occurs when $I = (X_1^d, X_2^d, \ldots, X_r^d)$, $r \leq n$. In this case, S_Δ is the coordinate ring of the toric variety in $\mathbb{P}^{rn-1}$ with parametric equations $\{Y_{ij} := X_i^d X_j\}$ where $i = 1, \ldots, r$ and $j = 1, \ldots, n$. Another interesting example is the case when $n = 3$ and I is the defining ideal of a set of points in $\mathbb{P}^2$ which is the intersection of two curves of the same degree d. A suitable embedding of the blowing up of $\mathbb{P}^2$ at these points yields a surface in $\mathbb{P}^5$ whose homogeneous coordinate ring is the diagonal subalgebra S_Δ of the Rees algebra S of I. Actually, the present work grew up from the desire to better understand the results of [10]. One is to believe that the algebraic approach via the diagonal of the Rees algebra may throw further light on the study not only of projective embeddings of rational surfaces obtained by blowing up a set of points in $\mathbb{P}^2$ (cf. [9, 11, 17]), but also of projective embeddings of rational n-folds obtained, more generally, by blowing up $\mathbb{P}^n$ along some special smooth subvariety.

In this section one first deals with the problem of computing the integral closure $\overline{(S_\Delta)}$ of S_Δ, thus obtaining that

$$\overline{(S_\Delta)} \simeq k[(\overline{I^s})_{s(d+1)} \mid s \geq 0] \cap k(\mathbf{X}I_d),$$

where $\overline{I^s}$ is the integral closure of I^s and $k(\mathbf{X}I)$ the field of fractions of the algebra $k[\mathbf{X}I_d]$. This ought to give a handy criterion, at least in the case of an ideal generated by monomials, of computing the integral closure of S_Δ, since the normalized powers $\overline{I^s}$ are within reach by the convex hull criterion.

The core of this section deals with the case where I is generated by a regular sequence of r homogeneous polynomials of the same degree. In this case one can establish an explicit presentation of S_Δ as well as compute its Hilbert function. The main result here says that S_Δ is a Cohen–Macaulay ring if $(r-1)d < n$, while failing to be so if $(r-1)d > n$.

The proof for the Cohen–Macaulayness is based on the aforementioned reduction to Segre products, while for the non-Cohen–Macaulayness one shows that the h-vector has a negative coefficient. Here, a crucial point is an appropriate formula for the Hilbert series.

As seen before, the case $r = 2$ is general enough to include some relevant geometric examples. If $r = 2$ and $n = 2, 3$, one sees that S_Δ is the homogeneous coordinate ring of a divisor on a rational normal scroll. Hence, by using a classical result of Buchsbaum and Eisenbud, it is possible to derive a minimal free resolution for S_Δ. This resolution has also been given by Holay in his thesis (see [18]) by methods which bear some relation to ours.

Looking at this resolution, one can see that S_Δ is Cohen–Macaulay if and only if $d \leq n$. For arbitrary $r \geq 2$ one is led to the conjecture that S_Δ is Cohen–Macaulay if and only if $(d-1)r \leq n$.

The remaining portion deals with the Cohen–Macaulayness of the diagonal subalgebra of $\mathscr{R}(I)$ for certain class of straightening closed ideals in polynomial algebras with straightening law, which includes the ideal generated by the maximal minors of a generic matrix. The proof makes heavy use of the "dévissage" to Segre products recorded earlier and of a substantial amount of combinatorics.

2. The diagonal subalgebra

Let $S = \bigoplus_{(u_1,\dots,u_n)\in\mathbb{Z}^n} S_{(u_1,\dots,u_n)}$ be a multigraded ring, where $S_{(u_1,\dots,u_n)}$ denotes the graded piece of S of degree $(u_1,\dots,u_n)$.

The central concept of this paper is the following.

Definition. The *diagonal subring* of S is the subring

$$S_\Delta := \bigoplus_{u\in\mathbb{Z}} S_{(u,\dots,u)}.$$

Clearly, S_Δ is a $\mathbb{Z}$-graded ring in a natural way. Also, if S is an algebra over a field k, S_Δ is a k-subalgebra of S.

The simplest case of a diagonal subalgebra occurs when $S = R_1 \otimes_k R_2$ is the tensor product of two graded k-algebras R_1 and R_2. Then S has a natural bigraded structure and its diagonal subalgebra S_Δ is the Segre product $R_1 \underline{\otimes}_k R_2$ of R_1 and R_2.

The classical situation has S a *standard bigraded algebra*, i.e. S is a bigraded k-algebra which admits a finite set of k-algebra generators of degrees $(1,0)$ and $(0,1)$. Then S_Δ is also standard graded. Say, if $S = k[x_1,\dots,x_n,t_1,\dots,t_r]$ for some elements x_i and t_j with $\deg x_i = (1,0)$ and $\deg t_i = (0,1)$, then

$$S_\Delta = k[x_it_j \mid 1 \le i \le n,\ 1 \le j \le r].$$

Geometrically, S stood for the bihomogeneous coordinate ring of a correspondence in the product $\mathbb{P}^{n-1} \times \mathbb{P}^{r-1}$ and S_Δ for the homogeneous coordinate ring of the image of the correspondence under the Segre embedding $\mathbb{P}^{n-1} \times \mathbb{P}^{r-1} \hookrightarrow \mathbb{P}^{nr-1}$.

2.1. Presentation

Henceforth we assume that S is a standard bigraded k-algebra; in this subsection we indicate how to get a presentation of the diagonal subalgebra S_Δ in terms of that of S.

Let us consider an algebra presentation $S \simeq A/J$ with $A = k[\mathbf{X},\mathbf{T}]$ a bigraded polynomial ring in two sets of mutually independent indeterminates $\mathbf{X}$ and $\mathbf{T}$ and J a bihomogeneous ideal of A, i.e. an ideal homogeneous, separately, in the $\mathbf{X}$-variables and in the $\mathbf{T}$-variables. Let $\mathbf{X} = \{X_1,\dots,X_n\}$ and $\mathbf{T} = \{T_1,\dots,T_r\}$. Then

$$A_\Delta = k[X_iT_j \mid 1 \le i \le n,\ 1 \le j \le r].$$

Let $\mathbf{U} = (U_{ij})$ be an $n \times r$ matrix of indeterminates. Mapping U_{ij} to X_iT_j yields a presentation:

$$A_\Delta \simeq k[\mathbf{U}]/I_2(\mathbf{U}),$$

where $I_2(\mathbf{U})$ denotes the ideal generated by the 2-minors of $\mathbf{U}$ ($I_2(\mathbf{U}) = 0$ if $r = 1$). By letting

$$J_\Delta := \bigoplus_{u \geq 0} J_{(u,u)},$$

it is clear that

$$S_\Delta = A_\Delta / J_\Delta.$$

To find the image of J_Δ in $k[\mathbf{U}]/I_2(\mathbf{U})$, one needs a set of generators of J_Δ.

Lemma 2.1. *Let $S \simeq A/J$ be a standard bigraded algebra as above. Suppose that J is generated by the homogeneous polynomials $F_1, \ldots, F_s$ with $\deg F_i = (a_i, b_i)$. Let $c_i = \max\{a_i, b_i\}$. Then J_Δ is generated by the elements of the form $F_i M$ where M is a monomial of degree $(c_i - a_i, c_i - b_i)$, $i = 1, \ldots, s$.*

Proof. Let f be an arbitrary element of J_Δ with $\deg f = (u,u)$. Then $f \in \sum_{i=1}^{s} F_i A_{(u-a_i, u-b_i)}$. Since A is generated by $A_{(1,0)}$ and $A_{(0,1)}$, one has

$$A_{(u-a_i,u-b_i)} = A_{(c_i-a_i,c_i-b_i)} A_{(u-c_i,u-c_i)}.$$

This yields the conclusion. □

Note that the monomials M of degree $(c_i - a_i, c_i - b_i)$ are monomials either in $\mathbf{X}$ with degree $c_i - a_i$ or in $\mathbf{T}$ with degree $c_i - b_i$, depending on whether c_i is equal to b_i or to a_i, respectively.

Let now F be a homogeneous element of J_Δ. First note that F is a linear combination of monomials of bidegree of the form (e,e), for some $e \geq 1$. Such a monomial can be expressed as a product of monomials of the form $X_i T_j$. Replacing all occurring products $X_i T_j$ by U_{ij} yields a preimage G of F in $k[\mathbf{U}]$. Note that $F \in J_\Delta$, we have several preimages, but they all coincide modulo $I_2(\mathbf{U})$. A presentation of S_Δ will be given by

$$S_\Delta \simeq k[\mathbf{U}]/\mathfrak{I}$$

where $\mathfrak{I}$ is the ideal of $k[\mathbf{U}]$ generated by $I_2(\mathbf{U})$ and a set of preimages of the generators of J_Δ.

Example 2.2. Let $S = k[\mathbf{X},\mathbf{T}]/I_2(V)$ where V is the matrix

$$\begin{pmatrix} X_1^d & \cdots & X_r^d \\ T_1 & \cdots & T_r \end{pmatrix}.$$

Then S is a presentation of the Rees algebra of the ideal $(X_1^d, \ldots, X_r^d) \subset k[\mathbf{X}] = k[X_1, \ldots, X_n]$, $r \leq n$. By Lemma 2.1, J_Δ is generated by the elements of the form $(X_i^d T_j - X_j^d T_i) M(\mathbf{T})$, $1 \leq i < j \leq r$, where $M(\mathbf{T})$ is a monomial of degree $d-1$ in $k[\mathbf{T}]$. By letting $\mathbf{U}_i := U_{i1}, \ldots, U_{ir}$, one sees that a preimage of such an element in $k[\mathbf{U}]$ is the element $U_{ij}M(\mathbf{U}_i) - U_{ji}M(\mathbf{U}_j)$. Therefore, $\mathfrak{I}$ is the ideal generated by $I_2(\mathbf{U})$ and these elements. In particular, if $n = r = 2$, then

$$S = k[X_1, X_2, T_1, T_2]/(X_2^d T_1 - X_1^d T_2),$$

and one has

$$S_\Delta = k[\mathbf{U}]/(U_{11}U_{22} - U_{12}U_{21}, U_{21}^d - U_{11}^{d-1}U_{12}, \\ U_{21}^{d-1}U_{22} - U_{11}^{d-2}U_{12}^2, \ldots, U_{21}U_{22}^{d-1} - U_{12}^d).$$

2.2. Dimension and multiplicity

Let S be a standard bigraded k-algebra. A bihomogeneous prime ideal $\wp$ of S is *relevant* if $\wp$ does not contain $S_{(1,0)}$ and $S_{(0,1)}$. Note that the biprojective spectrum $\mathrm{BiProj}\,(S)$ of S is the set of the relevant bihomogeneous prime ideals of S. It is easy to see that $\dim S/\wp \geq 2$ for any relevant prime ideal $\wp$. Following [19] we define

$$\mathrm{rel.dim}\, S := \begin{cases} 1 & \text{if } \mathrm{BiProj}\,(S) = \emptyset, \\ \max\{\dim S/\wp \mid \wp \in \mathrm{BiProj}\,(S)\} & \text{if } \mathrm{BiProj}\,(S) \neq \emptyset, \end{cases}$$

and call it the *relevant dimension* of S. Note that $\mathrm{rel.dim}\, S = \dim S$ if every associated prime of S is relevant. Let

$$H_S(u, v) := \dim_k S_{(u,v)}$$

be the Hilbert function of the bigraded algebra S.

It was proved by van der Waerden [25] that for large enough u and v

$$H_S(u, v) = \sum_{0 \leq i+j \leq \dim S - 2} a_{ij} \binom{u}{i} \binom{v}{j},$$

where a_{ij} are integers. This has been extended to the case when S is a standard bigraded algebra over an artinian ring by Bhattacharaya in [1]. Recently, Katz et al. [19, Theorem 2.2] showed that if $\mathrm{BiProj}\,(S) \neq \emptyset$, the total degree of the above polynomial is equal to $\mathrm{rel.dim}\, S - 2$ and $a_{ij} \geq 0$ for $i + j = \mathrm{rel.dim}\, S - 2$.

The result of Katz et al. still holds if $\mathrm{BiProj}\,(S) = \emptyset$. In this case, it is easy to check that $S_{(u,v)} = 0$ for u and v large enough, hence the above polynomial is zero and has degree -1.

We will denote the number a_{ij} by $e_S(i, j)$.

The Hilbert function of the diagonal subalgebra S_Δ can be expressed in terms of the Hilbert function of S as follows:

$$H_{S_\Delta}(u) = H_S(u, u).$$

Let $d = \text{rel.dim}\, S$. If u is large enough, one has

$$H_S(u,u) = \sum_{0 \le i+j \le d-2} e_S(i,j) \binom{u}{i} \binom{u}{j}$$

$$= \left(\sum_{i+j=d-2} \frac{e_S(i,j)}{i!j!} \right) u^{d-2} + \text{lower degree terms}$$

$$= \left[\frac{\sum_{i=0}^{d-2} e_S(i,d-2-i)\binom{d-2}{i}}{(d-2)!} \right] u^{d-2} + \text{lower degree terms.}$$

Proposition 2.3. *Let S be a standard bigraded k-algebra and $d = \text{rel.dim}\, S \ge 1$. Then*
(i) $\dim(S_\Delta) = d - 1$.
(ii) *If* $d \ge 2$, $e(S_\Delta) = \sum_{i=0}^{d-2} e_S(i, d-2-i)\binom{d-2}{i}$.

Proof. If $d = 1$, then $H_{S_\Delta}(u) = 0$ for u large enough, hence $\dim(S_\Delta) = 0$. Let $d \ge 2$. One needs to show that $d-2$ is the degree of the above polynomial. Since $\text{BiProj}(S) \ne \emptyset$, this follows from the facts that $e_S(i,d-2-i) \ge 0$ for $i = 0,\dots,d-2$ and that one of them is not zero. □

Example 2.4 (Fröberg and Hoa [8, Proposition 4]). Let R_1, R_2 be two standard graded k-algebras with dimension $\dim(R_1) = d_1$ and $\dim(R_2) = d_2$. Let S be the standard bigraded algebra $R_1 \otimes_k R_2$.

If $d_1 \ge 1$ and $d_2 \ge 1$, one has

$$H_S(u,v) = H_{R_1}(u) H_{R_2}(v)$$

$$= e(R_1)e(R_2) \binom{u}{d_1 - 1} \binom{v}{d_2 - 1} + \text{terms of total degree} < d_1 + d_2 - 2$$

for u and v large enough. Since $e(R_1) > 0$ and $e(R_2) > 0$, the degree of the above polynomial is $d_1 + d_2 - 2$. Hence $d = \text{rel.dim}\, S = d_1 + d_2 \ge 2$ and

$$e_S(i, d-2-i) = \begin{cases} 0 & \text{if } i \ne d_1 - 1, \\ e(R_1)e(R_2) & \text{if } i = d_1 - 1. \end{cases}$$

Since the Segre product $R_1 \underline{\otimes}_k R_2$ is isomorphic to S_Δ, we get
(i) $\dim R_1 \underline{\otimes}_k R_2 = d_1 + d_2 - 1$.
(ii) $e(R_1 \underline{\otimes}_k R_2) = e(R_1)e(R_2)\binom{d_1+d_2-2}{d_1-1}$.
If $d_1 = 0$ or $d_2 = 0$, then $\dim R_1 \underline{\otimes}_k R_2 = 0$.

Remark 2.5. (ii) gives a simple proof of the well-known formula for the multiplicity of the ideal $I_2(\mathbf{U})$ generated by the 2-minors of the $n \times r$ matrix $\mathbf{U}$ of indeterminates, since $k[\mathbf{U}]/I_2(\mathbf{U})$ is isomorphic to the Segre product of two polynomial rings R_1 and

R_2 with $\dim R_1 = n$ and $\dim R_2 = r$. Namely, one has

$$e(k[\mathbf{U}]/I_2(\mathbf{U})) = \binom{n+r-2}{n-1}.$$

2.3. Cohen–Macaulayness

An ancestor of the Cohen–Macaulayness of the diagonal subalgebra was taken up by Chow [7] who studied the problem for the Segre products of two Cohen–Macaulay standard graded algebras. This result was later improved by Stückrad–Vogel [23] for algebras of dimension ≥ 2. There are also other results on the Cohen–Macaulayness of Segre products in large classes of graded algebras [8, 15].

We will see that certain filtrations of a standard bigraded algebra may reduce the problem of the Cohen–Macaulayness of the diagonal subalgebra to the one of Segre products.

The following preliminaries are mainly borrowed from [14] (cf. also [20]).

For any filtration $\mathscr{F}$ of ideals of a commutative ring R, $\operatorname{gr}_{\mathscr{F}}(R)$ denotes the associated graded ring of R with respect to $\mathscr{F}$.

Let $S = A/J$ be a standard bigraded k-algebra, where $A = k[\mathbf{X},\mathbf{T}]$ is a polynomial ring in two sets of indeterminates $\mathbf{X}$ and $\mathbf{T}$ and J a bihomogeneous ideal of A. Set $\mathbf{T} = \{T_1,\ldots,T_r\}$. Consider an $\mathbb{N}^{r+1}$-gradation on A by setting

$$A_{(a_0,a_1,\ldots,a_r)} := k[\mathbf{X}]_{a_0} T_1^{a_1}\cdots T_r^{a_r}.$$

Let $\succcurlyeq$ be a term order on the monoid $\mathbb{N}^{r+1}$, i.e. a total order with the property

$$\mathbf{a} \succcurlyeq \mathbf{b} \quad \text{implies} \quad \mathbf{a}+\mathbf{c} \succcurlyeq \mathbf{b}+\mathbf{c} \quad \text{for all } \mathbf{c} \in \mathbb{N}^{r+1}.$$

Such a term order induces a filtration $\mathscr{F}$ on A with $\mathscr{F}_{\mathbf{a}} := \bigoplus_{\mathbf{b}\succcurlyeq\mathbf{a}} A_{\mathbf{b}}$. It is clear that $\operatorname{gr}_{\mathscr{F}}(A) \simeq A$. The filtration $\mathscr{F}$ imposes a filtration on S which we also denote by $\mathscr{F}$. Let J^* be the ideal generated by the initial forms of J, then

$$\operatorname{gr}_{\mathscr{F}}(S) \simeq A/J^*.$$

As a $\mathbb{N}^{r+1}$-graded ideal of A, J^* is also a bigraded ideal of A. Therefore, $\operatorname{gr}_{\mathscr{F}}(S)$ is a bigraded algebra, and, as such, has a diagonal subalgebra $\operatorname{gr}_{\mathscr{F}}(S)_\Delta$.

On the other hand, the above $\mathbb{N}^{r+1}$-gradation on A induces a $\mathbb{N}^r$-gradation on A_Δ by setting

$$(A_\Delta)_{(a_1,\ldots,a_r)} := A_{(a_1+\cdots+a_r,a_1,\ldots,a_r)}.$$

Note that this corresponds to an embedding of $\mathbb{N}^r$ in $\mathbb{N}^{r+1}$. Consider the restriction of the term order $\succcurlyeq$ on the additive monoid $\mathbb{N}^r$:

$$(a_1,\ldots,a_r) \succcurlyeq (b_1,\ldots,b_r) \quad \text{if} \quad (a_1+\cdots+a_r,a_1,\ldots,a_r) \succcurlyeq (b_1+\cdots+b_r,b_1,\ldots,b_r).$$

Similarly as above, this term order induces a filtration $\mathscr{F}_\Delta$ on A_Δ with $\operatorname{gr}_{\mathscr{F}_\Delta}(A_\Delta) \simeq A_\Delta$. $\mathscr{F}_\Delta$ will stand for the corresponding filtration on $S_\Delta = A_\Delta/J_\Delta$.

Proposition 2.6. *Let $S = A/J$ be a standard bigraded k-algebra and $\mathcal{F}$ and $\mathcal{F}_\Delta$ filtrations on S and S_Δ as above. Then*

$$\operatorname{gr}_{\mathcal{F}_\Delta}(S_\Delta) \simeq \operatorname{gr}_{\mathcal{F}}(S)_\Delta.$$

Proof. Let $(J_\Delta)^*$ be the ideal generated by the initial forms of the element of J_Δ with respect to the filtration $\mathcal{F}_\Delta$ on A_Δ. Then

$$\operatorname{gr}_{\mathcal{F}_\Delta}(S_\Delta) \simeq A_\Delta/(J_\Delta)^*.$$

On the other hand, one has

$$\operatorname{gr}_{\mathcal{F}}(S)_\Delta = (A/J^*)_\Delta = A_\Delta/(J^*)_\Delta.$$

Therefore, it suffices to show that $(J_\Delta)^* = (J^*)_\Delta$.

Let f be an arbitrary element of J_Δ and let f^* be the initial form of f with respect to $\mathcal{F}_\Delta$. Write $f = \sum f_{\mathbf{a}}$, where $f_{\mathbf{a}} \in J_{\mathbf{a}}$ and $\mathbf{a}$ is of the form $(a_1+\cdots+a_r, a_1, \ldots, a_r)$. Since $\mathcal{F}_\Delta$ comes from the term order $\succcurlyeq$ restricted on $\mathbb{N}^r$ by the embedding $(a_1, \ldots, a_r) \to (a_1 + \cdots + a_r, a_1, \ldots, a_r)$, one has $f^* = f_{\mathbf{b}}$ where $\mathbf{b} = \min\{\mathbf{a} \mid f_{\mathbf{a}} \neq 0\}$. But $f_{\mathbf{b}}$ is also the initial form of f with respect to the filtration $\mathcal{F}$ on A. Therefore, $f^* \in J^* \cap A_\Delta = (J^*)_\Delta$. This proves that $(J_\Delta)^* \subseteq (J^*)_\Delta$.

For the converse, let g be an arbitrary homogeneous element of $(J^*)_\Delta$. Then $g \in A_\Delta$ and g is the initial form of an element $h \in J$ with respect to $\mathcal{F}$. Write $h = \sum f_{(u,v)}$ with $f_{(u,v)} \in J_{(u,v)}$. Since the $\mathbb{N}^{r+1}$-gradation of A is finer than the original $\mathbb{N}^2$-gradation of A, g is the initial form of some element $f_{(u,v)}$ with respect to $\mathcal{F}$. It follows that $\deg g = (u, v)$. Since $g \in A_\Delta$, $u = v$. Hence $f_{(u,v)} \in J_\Delta$. As we have seen above, g is also the initial form of $f_{(u,v)}$ with respect to the filtration $\mathcal{F}_\Delta$ on A_Δ. Therefore, $g \in (J_\Delta)^*$. □

Corollary 2.7. *If $\operatorname{gr}_{\mathcal{F}}(S)_\Delta$ is a Cohen–Macaulay ring then S_Δ is a Cohen–Macaulay ring.*

Proof. It is well known that S_Δ is Cohen–Macaulay as soon as the associated graded ring $\operatorname{gr}_{\mathcal{F}_\Delta}(S_\Delta)$ is Cohen–Macaulay. □

We next explain the rough strategy of reduction to Segre products.

By the definition of the filtration $\mathcal{F}$, the initial form of any element of A is the product fM of a homogeneous polynomial f of $k[\mathbf{X}]$ and a monomial M of $k[\mathbf{T}]$. Since the ideal J^* is generated by such polynomials, J^* is a finite intersection of ideals of the form $(I, T_1^{a_1}, \ldots, T_r^{a_r})$ where I is a homogeneous ideal of $k[\mathbf{X}]$. Passing to the diagonal, one has

$$(J^*)_\Delta = \bigcap (I, T_1^{a_1}, \ldots, T_r^{a_r})_\Delta.$$

Since $A/(I, T_1^{a_1}, \ldots, T_r^{a_r}) \simeq k[\mathbf{X}]/I \otimes_k k[\mathbf{T}]/(T_1^{a_1}, \ldots, T_r^{a_r})$, its diagonal subalgebra $A_\Delta/(I, T_1^{a_1}, \ldots, T_r^{a_r})_\Delta$ is isomorphic to the Segre product $k[\mathbf{X}]/I \underline{\otimes}_k k[\mathbf{T}]/(T_1^{a_1}, \ldots, T_r^{a_r})$ in which case one may apply the existing Cohen–Macaulayness criteria.

Finally, to go back to $\mathrm{gr}_{\mathcal{F}}(S)_\Delta \simeq A_\Delta/(J^*)_\Delta$, one may use the Cohen–Macaulay *dévissage* invented by Eagon–Hochster [16], which we recall here in a modified form:

Lemma 2.8. *Let R be a standard graded algebra and let Q_j, $1 \le j \le s$, be ideals of R satisfying the following properties:*
(i) *R/Q_j is a Cohen–Macaulay ring of dimension d for $j = 1,\dots,s$.*
(ii) *$R/(Q_1 \cap \cdots \cap Q_j + Q_{j+1})$ is a Cohen–Macaulay ring of dimension $d-1$ for $j = 1,\dots,s-1$.*
Then R/Q is a Cohen–Macaulay ring, where $Q = \bigcap_{j=1}^s Q_j$.

Proof. Using induction we can reduce to the case $s = 2$. For this case, the statement already follows from [16, Proposition 18]. □

It should be noted that in the case of diagonal subalgebras, when $Q = (J^*)_\Delta \in R = A_\Delta$ and Q_j is of the form $(I, T_1^{a_1},\dots,T_r^{a_r})_\Delta$, then $R/(Q_1 \cap \cdots \cap Q_j + Q_{j+1})$ is the diagonal subalgebra of a graded algebra A/H where H is an ideal generated by polynomials of the form fM with $f \in k[\mathbf{X}]$ and M a monomial of $k[\mathbf{T}]$ and, as such, has a decomposition into ideals of the form $(I, T_1^{a_1},\dots,T_r^{a_r})$ again.

Surprisingly enough, one can actually carry out the above steps in some cases, such as for the presentation ideals of the Rees algebras of ideals generated by regular sequences [14] or by straightening closed ideals in the poset of an algebra with straightening law [20], where a good hold of the initial ideals is within reach. This will be done in the next section.

2.4. Integral closure and normality

In this part we will consider the integral closure $\overline{S}$ of a multigraded domain S and relate $\overline{S}_\Delta$ to the integral closure of S_Δ. In the following $K(S)$ will denote the field of fraction of the multigraded domain S. We first prove a basic result.

Lemma 2.9. *Let S be a $\mathbb{Z}^n$-graded domain and let M the multiplicative set of non-zero homogeneous elements of S. Then*
(i) *The ring of fractions S_M is an integrally closed $\mathbb{Z}^n$-graded domain.*
(ii) *$\overline{S}$ is a $\mathbb{Z}^n$-graded subring of S_M, which is $\mathbb{N}^n$-graded if S is $\mathbb{N}^n$-graded.*

Proof. It is clear that S_M has a natural structure of $\mathbb{Z}^n$-graded ring by letting $(S_M)_{(u_1,\dots,u_n)}$ to be the set of elements of the form s/t where $s \in S$, $t \in M$ are homogeneous elements such that $\deg(s) - \deg(t) = (u_1,\dots,u_n)$. It is also clear that $S \subset S_T \subset K(S)$. Following [3], where the case of $\mathbb{Z}$-graded domain is considered, one can easily prove that S_M is integrally closed. Hence we get $\overline{S} \subset S_M$. By degree reasoning, every homogeneous summand of an element of $\overline{S}$ belongs to $\overline{S}$ again. From this it follows that $\overline{S}$ is a $\mathbb{Z}^n$-graded subring of S_M. Finally, since S is a domain, if S is $\mathbb{N}^n$ multigraded, so is $\overline{S}$. □

Proposition 2.10. *Let S be a multigraded domain. Then*

$$\overline{(S_\Delta)} = \overline{S}_\Delta \cap K(S_\Delta).$$

Proof. By Lemma 2.9, $\overline{(S_\Delta)}$ is a graded ring and every homogeneous element f of $\overline{(S_\Delta)}$ is a fraction of two homogeneous elements of S_Δ. It follows that $f \in (S_M)_\Delta$. Since f is also integral over S, we get $f \in \overline{S} \cap (S_M)_\Delta = \overline{S}_\Delta$. So we have proved that $\overline{(S_\Delta)} \subseteq \overline{S}_\Delta \cap K(S_\Delta)$.

For the converse, let $f = \sum f_u$ be an element of $\overline{S}_\Delta \cap K(S_\Delta)$. Since $f \in \overline{S}_\Delta \subset \overline{S}$ and $\overline{S}$ is $\mathbb{Z}^n$-graded, f_u is integral over S for every u. This implies that any relation of integral dependence for f_u over S is one for f_u over S_Δ. Hence f_u is integral over S_Δ for every u, so that f is integral over S_Δ. Since $f \in K(S_\Delta)$, one has $f \in \overline{(S_\Delta)}$ as wanted. □

Corollary 2.11. *If the multigraded ring S is a normal domain, then so is its diagonal subring S_Δ.*

This result can also be proved using the following notion which is of independent interest.

A *Reynolds operator* of a ring extension $D \subset C$ is a D-module surjection $\varphi : B \to A$ such that the composite $D \subset C \xrightarrow{\varphi} D$ is the identity map. Clearly, a Reynolds operator exists if and only if D is a direct summand of C as D-modules (in which case, the corresponding projector is a Reynolds operator). By [17, Proposition 6.15] normality carries from C to D in a ring extension $D \subset C$ which has a Reynolds operator. Therefore, Corollary 2.11 is a consequence of the following result which holds in a more general setup.

Proposition 2.12. *Let S be a multigraded k-algebra. Then the ring extension $S_\Delta \subset S$ has a Reynolds operator.*

Proof. Consider the subset $S' = \sum_{u \notin \Delta} S_{(u_1,\dots,u_n)} \subset S$. Clearly, S' is a k-subspace of S and as such admits S_Δ as a direct complement. However, S' is actually an S_Δ-submodule of S, so one is done. □

3. Case study: Rees algebras

We will be concerned with the Rees algebra

$$\mathscr{R}(I) := \bigoplus_{s \geq 0} I^s t^s$$

of a homogeneous ideal I generated by forms $f_1, \dots, f_r$ of a fixed degree d in a polynomial ring $k[\mathbf{X}] = k[X_1, \dots, X_n]$. It is clear that

$$\mathscr{R}(I) = k[\mathbf{X}][f_1 t, \dots, f_r t] \subset k[\mathbf{X}, t]$$

and as a subring of the bigraded algebra $k[\mathbf{X},t]$, $\mathscr{R}(I)$ is naturally bigraded. Since the elements f_j have the same degree, if we set $\deg X_i = (1,0)$ and $\deg f_jt = (0,1)$, then $\mathscr{R}(I)$ becomes a standard bigraded k-algebra. Let S denote this standard bigraded algebra. Then S is a domain and $\mathrm{rel.dim}\, S = \dim S = n+1$.

One sees immediate to see that

$$S_{(u,s)} = (I^s)_{u+sd}t^s$$

and, as a k-vector space, this is generated by elements of the form MN where M is a monomial of $k[\mathbf{X}]$ with degree u and N a product of s copies of f_jt. Therefore one has a neat description of the diagonal subalgebra of S, namely, $S_\Delta = \bigoplus_{s\geq 0}(I^s)_{s(d+1)}t^s$, from which, it is clear that

$$S_\Delta = k[X_if_jt \mid 1\leq i\leq n,\ 1\leq j\leq r] \simeq k[X_if_j \mid 1\leq i\leq n,\ 1\leq j\leq r].$$

In the sequel, we will thus stick to the notation $k[\mathbf{X}I_d]$ instead of S_Δ. Note that $k[\mathbf{X}I_d]$ is the special fiber of the Rees algebra of the ideal $(\mathbf{X})I$. Finally, by Proposition 2.3, $\dim S_\Delta = n$.

3.1. Integral closure

Let $k(\mathbf{X}I_d)$ denote the field of fractions of $k[\mathbf{X}I_d]$. One has $\mathrm{Proj}\,(S_\Delta) = \mathrm{Proj}\,(S)$. Therefore, $k(\mathbf{X}I_d)$ is a purely transcendental extension of k with $\mathrm{tr.deg}_k\, k(\mathbf{X}I_d) = \dim k[\mathbf{X}] = n$. A simple calculation yields a little more, as follows.

Proposition 3.1. (i) $k(\mathbf{X}I_d) = k(X_1/X_i,\ldots,X_n/X_i)(X_if_j)$ *for any fixed choice of indices* $1\leq i\leq n$ *and* $1\leq j\leq r$.

(ii) *The extension* $k(\mathbf{X}) \mid k(\mathbf{X}I_d)$ *is simple algebraic of degree* $d+1$, *generated by any chosen* X_i.

(iii) *The ring extension* $k[\mathbf{X}I_d]\subset k(\mathbf{X})$ *is integral if and only if the ideal* I *is* $(\mathbf{X})$-*primary.*

Proof. (i) Since $X_l/X_i = X_lf_1/X_if_1$, for any l, $1\leq l\leq n$, we have an obvious inclusion $k(X_1/X_i,\ldots,X_n/X_i)(X_if_j)\subset k(\mathbf{X}I_d)$. The reverse inclusion follows from the easy equalities

$$\begin{cases} X_lf_j = (X_l/X_i)X_if_j & \in k(X_1/X_i,\ldots,X_n/X_i)(X_if_j) \\ X_lf_k = X_lf_j(f_k/f_j) & \in k(X_1/X_i,\ldots,X_n/X_i)(X_if_j)(f_k/f_j) \end{cases}$$

and from the fact that $f_k/f_j\in k(X_1/X_i,\ldots,X_n/X_i)$ since f_k, f_j are homogeneous of the same degree.

(ii) Write $f_j = X_i^d f_j(X_1/X_i,\ldots,X_n/X_i)$. Consider $\alpha = f_j(X_1/X_i,\ldots,X_n/X_i)$ as an element of the subfield $k(X_1/X_i,\ldots,X_n/X_i)$. Then, X_i satisfies the algebraic equation $Y^{d+1} - \alpha^{-1}(X_if_j)$ over the field $k(\mathbf{X}I_d)$. Since $X_l = (X_l/X_i)X_i$, for any other index $l, 1\leq i\leq n$, the claim on the generation of $k(\mathbf{X}) \mid k(\mathbf{X}I_d)$ is shown. To see that the degree is exactly $d+1$, one observes that, by part (iii) below – whose proof is independent of degree considerations – the element $X_i f_j$ is transcendental over the subfield $k(X_1/X_i,\dots,X_n/X_i)$, hence cannot factor in $k(\mathbf{X}I_d)$.

(iii) If I is $(\mathbf{X})$-primary then I must contain powers of all variables $\mathbf{X}$, so $k[\mathbf{X}]$ is obviously integral over $k[\mathbf{X}I_d]$. Conversely, if X_i satisfies an equation of integral dependence over $k[\mathbf{X}I_d]$ then, by reasoning with degrees, one sees that some $X_i f_j$, hence f_j itself, must be a power of X_i. □

In order to compute the integral closure of S_Δ we use the normalized Rees algebra

$$\mathscr{R}_{\mathrm{n}}(I) := \bigoplus_{s\geq 0} \overline{I^s} t^s,$$

where $\overline{I^s}$ denotes the integral closure of the ideal I^s. It is well known that $\mathscr{R}_{\mathrm{n}}(I)$ is the integral closure of the Rees algebra S. As such, it inherits a bigraded structure from S. We can also describe this graded structure.

Lemma 3.2. $\mathscr{R}_{\mathrm{n}}(I) = \bigoplus_{(u,s)\in\mathbb{N}^2} \left(\overline{I^s}\right)_{u+sd} t^s$.

Proof. According to Lemma 2.9 an homogeneous element of $\overline{S}$ of degree (u,s) is a fraction a/b where $a\in S_{(p,q)}$, $b\in S_{(v,w)}$ and

$$p-v=u, \qquad q-w=s.$$

Hence one can write $a/b=(ct^q)/(et^w)$ where $c\in(I^q)_{p+qd}$, $e\in(I^w)_{v+wd}$. Thus

$$a/b=(c/e)t^s\in\overline{S}=\bigoplus_{s\geq 0}\overline{I^s}t^s$$

and one can write $(c/e)=h$ for some $h\in\overline{I^s}$. But then $c=he$ implies that h is an homogeneous element in $\overline{I^s}$ of degree $p+qd-(v+wd)=u+sd$. □

As a corollary one gets the main result of this part, namely, the following explicit description of the integral closure $\overline{(S_\Delta)}$ of S_Δ.

Theorem 3.3. *Let $I\subset k[\mathbf{X}]$ be an ideal generated by homogeneous polynomials of fixed degree d. Let $S=\mathscr{R}(I)$ be the standard bigraded Rees algebra of I. Then*

$$\overline{(S_\Delta)}\simeq k[\,(\overline{I^s})_{s(d+1)}\,|\,s\geq 0]\cap k(\mathbf{X}I_d).$$

Proof. By Proposition 2.10

$$\overline{(S_\Delta)}=\mathscr{R}_{\mathrm{n}}(I)_\Delta\cap K(S_\Delta),$$

where $K(S_\Delta)$ is the field of fraction of S_Δ. As seen at the beginning of this section, $K(S_\Delta)\simeq k(\mathbf{X}I_d)$. By Lemma 3.2, this isomorphism induces an isomorphism

$$\mathscr{R}_{\mathrm{n}}(I)_\Delta \simeq k[(\overline{I^s})_{s(d+1)} \,|\, s \geq 0],$$

and one is done. □

Theorem 3.3 gives a handy formula for the computation of the integral closure of the diagonal subalgebra of $S = \mathscr{R}(I)$.

Example 3.4. Let $I = (X_1^d, \dots, X_r^d) \subset k[\mathbf{X}] = k[X_1, \dots, X_n]$. Then

$$S_\Delta \simeq k[\mathbf{X}I_d] = k[X_iX_j^d \,|\, 1 \leq i \leq n,\ 1 \leq j \leq r].$$

To compute the integral closure of this algebra one has to compute $(\overline{I^s})_{s(d+1)}$ for all $s \geq 1$. It is clear that $\overline{I} = (X_1, \dots, X_r)^d$. From this it follows that $\overline{I^s} = (X_1, \dots, X_r)^{sd}$. The vector space $(X_1, \dots, X_r)^{sd}_{s(d+1)}$ is generated by elements of the form MN with M a monomial of degree s in $X_1, \dots, X_n$ and N a monomial of degree sd in $X_1, \dots, X_r$. It is clear that MN can be rewritten as a product of s monomials of the form X_iL, $1 \leq i \leq n$, where L is a monomial of degree d in $X_1, \dots, X_r$. Hence

$$k[(\overline{I^s})_{s(d+1)} \,|\, s \geq 0] = k[X_iL \,|\, 1 \leq i \leq n,\ L \text{ a monomial of degree } d \text{ in } X_1, \dots, X_r].$$

Since $X_iL = (X_iL/X_1^{d+1})X_1^{d+1} \in k(X_2/X_1, \dots, X_n/X_1)(X_1^{d+1}) = k(\mathbf{X}I_d)$, it is clear that

$$k[X_iL \,|\, 1 \leq i \leq n,\ L \text{ a monomial of degree } d \text{ in } X_1, \dots, X_r] \subset k(\mathbf{X}I_d).$$

Therefore, applying Theorem 3.3 one gets

$$\begin{aligned}\overline{(S_\Delta)} &\simeq k[(\overline{I^s})_{s(d+1)} \,|\, s \geq 0] \cap k(\mathbf{X}I_d) \\ &\simeq k[X_iL \,|\, 1 \leq i \leq n,\ L \text{ a monomial of degree } d \text{ in } X_1, \dots, X_r].\end{aligned}$$

3.2. Complete intersections

Throughout this part, let $I = (f_1, \dots, f_r)$ be an ideal generated by a regular sequence of r forms in $k[\mathbf{X}] = k[X_1, \dots, X_n]$ of fixed degree d.

As before, set $\mathbf{T} = \{T_1, \dots, T_r\}$ and $A = k[\mathbf{X}, \mathbf{T}]$. It is well known that the Rees algebra $S = \mathscr{R}(I)$ has the presentation $S = A/J$, where J is the ideal generated by the elements $F_{ij} = f_iT_j - f_jT_i$, $1 \leq i < j \leq r$ of bidegree $(d, 1)$. According to Lemma 2.1, J_Δ is generated by the elements $F_{ij}M$, where M runs through the monomials of degree $d-1$ in $T_1, \dots, T_r$.

Proposition 3.5. *Let* $\mathbf{U} = (U_{ij})$ *be an* $n \times r$ *matrix of indeterminates. A presentation of* S_Δ *is given by*

$$S_\Delta \simeq k[\mathbf{U}]/\mathfrak{I},$$

where $\Im$ is the ideal generated by the $\binom{n}{2}\binom{r}{2}$ 2-minors of $\mathbf{U}$ and by the preimages of the generators of J_Δ under the map $U_{ij} \to X_iT_j$. Further, if $d \geq 2$, a minimal set of generators of $\Im$ consists of $\binom{n}{2}\binom{r}{2}$ forms of degree two and $d\binom{d+r-1}{r-2}$ forms of degree d.

Proof. The presentation of S_Δ follows from the description given in Subsection 2.1. Let $d \geq 2$. Since the preimages of $F_{ij}M$ have degree d, in order to prove the second statement, it suffices to show that

$$\dim_k(\Im/I_2(\mathbf{U}))_d = d\binom{d+r-1}{r-2},$$

i.e., that

$$H_{A_\Delta}(d) - H_{S_\Delta}(d) = d\binom{d+r-1}{r-2}.$$

Now, since A is the Segre product of $k[\mathbf{X}]$ and $k[\mathbf{T}]$, one has

$$H_{A_\Delta}(d) = \binom{d+n-1}{n-1}\binom{d+r-1}{r-1}.$$

The Hilbert function of S_Δ will be computed in the next theorem, where it will be seen that, with $R = k[\mathbf{X}]$,

$$\begin{aligned} H_{S_\Delta}(d) &= \sum_{j=1}^{r}(H_R(d) - j + 1)\binom{d+j-2}{j-1} \\ &= H_R(d)\sum_{j=1}^{r}\binom{d+j-2}{j-1} - d\sum_{j=1}^{r}\binom{d+j-2}{j-2} \\ &= \binom{d+n-1}{n-1}\binom{d+r-1}{r-1} - d\binom{d+r-1}{r-2}. \end{aligned}$$

The conclusion follows. □

Going back to the notation of Subsection 2.3, let $A = k[\mathbf{X},\mathbf{T}]$ be $\mathbb{N}^{r+1}$-graded by setting

$$A_{(a_0,a_1,\ldots,a_r)} := k[\mathbf{X}]_{a_0}T_1^{a_1}\cdots T_r^{a_r}.$$

The degree lexicographic term order on $\mathbb{N}^{r+1}$ induces a filtration $\mathscr{F}$ on A. As seen earlier, $\mathscr{F}$ imposes a filtration on S such that $\operatorname{gr}_{\mathscr{F}}(S) \simeq A/J^*$, where J^* denotes the ideal generated by the initial forms of the elements of J. It has been shown in [14] that

$$J^* = (I_2T_2, \ldots, I_rT_r),$$

where $I_j := (F_1, \ldots, F_{j-1})$, $j = 2, \ldots, r$. We will use this description of J^* to compute the Hilbert function $H_{S_\Delta}(u)$, the Hilbert series $P_{S_\Delta}(z)$ and to study the Cohen–Macaulayness of S_Δ.

Theorem 3.6. *Let $I \subset k[X_1,\dots,X_n]$ be an ideal generated by a regular sequence of forms of degree d. Let $S = \mathscr{R}(I)$ be the standard bigraded Rees algebra of I. Then*

(i) $H_{S_\Delta}(u) = \sum_{i=0}^{r-1}(-1)^i\binom{u-id+n-1}{n-1}\binom{u+r-1}{r-i-1}\binom{u+i-1}{i}$ *for* $u \geq 0$.

(ii) $e(S_\Delta) = \sum_{i=0}^{r-1} d^i\binom{n-1}{i}$.

(iii) $P_{S_\Delta}(z) = \frac{1}{(1-z)^n} + \sum_{j=2}^{r}\frac{z}{(j-1)!}\frac{\mathrm{d}^{j-1}}{\mathrm{d}z^{j-1}}\left[\frac{z^{j-2}(1-z^d)^{j-1}}{(1-z)^n}\right]$.

Proof. (i) Any element $f \in J^*$ with $\deg f = (a_0,a_1,\dots,a_r)$ is of the form $fT_1^{a_1}\cdots T_r^{a_r}$ with $f \in (I_j)_{a_0}$ for some element $j = 2,\dots,r$ with $a_j \neq 0$. Since I_j is an increasing sequence of ideals, setting $|(a_1,\dots,a_r)| := \max\{j \mid a_j \neq 0\}$, one has

$$(J^*)_{(a_0,a_1,\dots,a_r)} = (I_{|(a_1,\dots,a_r)|})_{a_0}T_1^{a_1}\cdots T_r^{a_r}.$$

Therefore, the Hilbert function of $\mathrm{gr}_{\mathscr{F}}(S)$ as a $\mathbb{N}^{r+1}$-graded algebra is given by

$$H_{\mathrm{gr}_{\mathscr{F}}(S)}(a_0,a_1,\dots,a_r) = H_{R/I_{|(a_1,\dots,a_r)|}}(a_0)$$

where $R := k[\mathbf{X}]$. From this, one gets the Hilbert function of $\mathrm{gr}_{\mathscr{F}}(S)$ as a bigraded algebra:

$$H_{\mathrm{gr}_{\mathscr{F}}(S)}(u,v) = \sum_{a_1+\cdots+a_r=v} H_{R/I_{|(a_1,\dots,a_r)|}}(u) = \sum_{j=1}^{r} H_{R/I_j}(u)\binom{v+j-2}{j-1},$$

where the latter equality follows from the fact that the number of vectors $(a_1,\dots,a_r)$ such that $a_1+\cdots+a_r = v$ and $|(a_1,\dots,a_r)| = j$ is given by $\binom{v+j-2}{j-1}$. Since I_j is generated by a regular sequence of $j-1$ forms of degree d, one has

$$H_{R/I_j}(u) = \sum_{i=0}^{j-1}(-1)^i\binom{j-1}{i}H_R(u-id) = \sum_{i=0}^{j-1}(-1)^i\binom{j-1}{i}\binom{u-id+n-1}{n-1}.$$

It follows that

$$\begin{aligned}
H_{\mathrm{gr}_{\mathscr{F}}(S)}(u,v) &= \sum_{j=1}^{r}\sum_{i=0}^{j-1}(-1)^i\binom{j-1}{i}\binom{u-id+n-1}{n-1}\binom{v+j-2}{j-1}\\
&= \sum_{i=0}^{r-1}(-1)^i\binom{u-id+n-1}{n-1}\sum_{j=i+1}^{r}\binom{j-1}{i}\binom{v+j-2}{j-1}\\
&= \sum_{i=0}^{r-1}(-1)^i\binom{u-id+n-1}{n-1}\sum_{j=i+1}^{r}\binom{v+j-2}{j-i-1}\binom{v+i-1}{i}\\
&= \sum_{i=0}^{r-1}(-1)^i\binom{u-id+n-1}{n-1}\binom{v+r-1}{r-i-1}\binom{v+i-1}{i}.
\end{aligned}$$

Since $H_S(u,v) = H_{\mathrm{gr}_{\mathscr{F}}(S)}(u,v)$ and $H_{S_\Delta}(u) = H_S(u,u)$, setting $v = u$ in the above formula yields the desired formula for $H_{S_\Delta}(u)$.

(ii) Note that $\dim R/I_j = n - j + 1$ and $e(R/I_j) = d^{j-1}$. Then one has

$$H_{R/I_j}(u) = \frac{d^{j-1}}{(n-j)!}u^{n-j} + \text{terms of lower degree.}$$

It follows that

$$H_S(u,v) = \sum_{j=1}^{r} H_{R/I_j}(u)\binom{v+j-2}{j-1}$$
$$= \sum_{j=1}^{r} \frac{d^{j-1}}{(n-j)!(j-1)!}u^{n-j}v^{j-1} + \text{terms of degree} < n-1.$$

In the notation of Subsection 2.2, this means that

$$e_S(j, n-j-1) = \begin{cases} d^{n-j-1} & \text{if } n-r \le j \le n-1, \\ 0 & \text{if } 0 \le j \le n-r-1. \end{cases}$$

Then, by Proposition 2.3,

$$e(S_\Delta) = \sum_{j=0}^{n-1} e_S(j, n-j-1)\binom{n-1}{j} = \sum_{j=n-r}^{n-1} d^{n-j-1}\binom{n-1}{j} = \sum_{i=0}^{r-1} d^i\binom{n-1}{i}.$$

(iii) The Hilbert series is now straightforward to obtain:

$$P_{S_\Delta}(z) = \sum_{u\ge 0}\sum_{j=1}^{r} H_{R/I_j}(u)\binom{u+j-2}{j-1}z^u$$
$$= \sum_{u\ge 0} H_R(u) + \sum_{u\ge 0}\sum_{j=2}^{r}\frac{z}{(j-1)!}\frac{\mathrm{d}^{j-1}}{\mathrm{d}z^{j-1}}[H_{R/I_j}(u)z^{u+j-2}]$$
$$= \frac{1}{(1-z)^n} + \sum_{j=2}^{r}\frac{z}{(j-1)!}\frac{\mathrm{d}^{j-1}}{\mathrm{d}z^{j-1}}[z^{j-2}P_{R/I_j}(z)]$$
$$= \frac{1}{(1-z)^n} + \sum_{j=2}^{r}\frac{z}{(j-1)!}\frac{\mathrm{d}^{j-1}}{\mathrm{d}z^{j-1}}\left[\frac{z^{j-2}(1-z^d)^{j-1}}{(1-z)^n}\right]. \qquad \square$$

Theorem 3.7. *Let $I = (f_1,\ldots,f_r) \subset k[X_1,\ldots,X_n]$ be an ideal generated by a regular sequence of r forms of degree d. Let $S = \mathscr{R}(I)$ be the standard bigraded Rees algebra of I. Then*:

(i) *S_Δ is a Cohen–Macaulay ring if $(r-1)d < n$.*

(ii) *S_Δ is not a Cohen–Macaulay ring if $(r-1)d > n$.*

Proof. (i) If $r = n$, the condition $(r-1)d < n$ implies $d = 1$. In this case, $I = (X_1,\ldots,X_n)$. Then $S_\Delta \simeq k[I_2]$ is isomorphic to the coordinate ring of the Veronese embedding of $\mathbb{P}^{n-1}$ in $\mathbb{P}^{\binom{n+2}{2}-1}$, which is known to be Cohen–Macaulay.

Let $r < n$. By Corollary 2.7, S_Δ is Cohen–Macaulay if $\mathrm{gr}_{\mathcal{F}}(S)_\Delta = A_\Delta/(J^*)_\Delta$ is Cohen–Macaulay. Recall that $J^* = (I_2T_2, \dots, I_rT_r)$ with $I_j = (f_1, \dots, f_{j-1})$, $j = 1, \dots, r$. Set

$$Q_j = (I_j, T_{j+1}, \dots, T_r)$$

with the proviso $I_0 = 0$ and $T_{r+1} = 0$. It is easily seen that $J^* = \bigcap_{j=1}^r Q_j$ (cf. [14]). Passing to the diagonal gives

$$(J^*)_\Delta = \bigcap_{j=1}^{r} (Q_j)_\Delta.$$

It suffices to show that this decomposition satisfies the conditions of Lemma 2.8. As pointed out in Subsection 2.3, $A_\Delta/(Q_j)_\Delta \simeq k[\mathbf{X}]/I_j \underline{\otimes}_k k[\mathbf{T}]/(T_{j+1}, \dots, T_r)$, which is the Segre product of two homogeneous complete intersections. Since $I_1 = 0$, $A_\Delta/(Q_1)_\Delta \simeq k[\mathbf{X}] \underline{\otimes}_k k[T_1] \simeq k[\mathbf{X}]$. If $j > 1$, the complete intersections have dimension ≥ 2 (one needs $r < n$ for $j = r$). In [23, Corollary 1], Stückrad and Vogel already gave a criterion for the Cohen–Macaulayness of such a Segre product in terms of the degrees of the generators of the complete intersections. Applying this criterion it is easy to check that $A_\Delta/(Q_j)_\Delta$ is Cohen–Macaulay if $(j-1)d \leq n-1$. By Example 2.4 we have

$$\begin{aligned}\dim A_\Delta/(Q_j)_\Delta &= \dim k[\mathbf{X}]/(f_1, \dots, f_{j-1}) + \dim k[\mathbf{T}]/(T_{j+1}, \dots, T_r) - 1 \\ &= (n - j + 1) + j - 1 = n.\end{aligned}$$

It remains to show that $A_\Delta/((Q_1)_\Delta \cap \cdots \cap (Q_j)_\Delta + (Q_{j+1})_\Delta)$ is a Cohen–Macaulay ring with dimension $n-1$ for $j = 1, \dots, r-1$. Since

$$Q_1 \cap \cdots \cap Q_j + Q_{j+1} = (I_{j+1}, T_{j+1}, \dots, T_r),$$

this can be proved similarly as above.

(ii) As is well-known, in order to show that S_Δ is *not* Cohen–Macaulay, it suffices to detect a negative coefficient in the h-vector of S_Δ, i.e., in the numerator of the Hilbert series $P_{S_\Delta}(z) = h(z)/(1-z)^n$, $h(1) \neq 0$.

Now, Theorem 3.6 (iii) yields

$$h(z) = 1 + \sum_{j=2}^{r} \frac{z}{(j-1)!} h_j(z),$$

where

$$h_j(z) := (1-z)^n \frac{\mathrm{d}^{j-1}}{\mathrm{d}z^{j-1}} \left[\frac{z^{j-2}(1-z^d)^{j-1}}{(1-z)^n} \right].$$

Let us compute the coefficients of $z^{(r-1)d}$ and $z^{(r-1)d-1}$ in $h(z)$. If $(r-1)d > n$, then $d \geq 2$ and $r \geq 2$. Now, $\deg(h_j(z)) \leq (j-1)d - 1$ for every $j = 2, \dots, r$. Therefore these integers coincide with the coefficients of $z^{(r-1)d-1}$ and $z^{(r-1)d-2}$ in $h_r(z)$, divided by $(r-1)!$.

Let $a := r - 2 + (r-1)d - n$; it is easy to see that

$$g(z) := \frac{z^{r-2}(1-z^d)^{r-1}}{(1-z)^n} + (-1)^{r-n}(z^a + nz^{a-1})$$

has degree $\le a-2$. Hence one can write

$$\frac{z^{r-2}(1-z^d)^{r-1}}{(1-z)^n} = (-1)^{r-n+1}(z^a + nz^{a-1}) + g(z),$$

where $g(z)$ has degree $\le a-2$. It follows that

$$h_r(z) = (-1)^{r-n+1}(1-z)^n[a\cdots(a-r+2)z^{a-r+1} + n(a-1)\cdots(a-r+1)z^{a-r}]$$
$$+ \text{ terms of degree } < n+a-r.$$

It follows that the coefficient of $z^{(r-1)d-1} = z^{a-r+1+n}$ is

$$(-1)^{r+1}a(a-1)\ldots(a-r+2),$$

while that of $z^{(r-1)d-2} = z^{a-r+n}$ is

$$(-1)^{r+1}[n(a-1)\ldots(a-r+1) - na\ldots(a-r+2)]$$
$$= (-1)^r n(r-1)(a-1)\ldots(a-r+2).$$

Since $(r-1)d > n$, one has $r \ge 2$ and $a > r-2$. Hence, if r is even

$$(-1)^{r+1}a(a-1)\ldots(a-r+2) < 0;$$

if r is odd

$$(-1)^r n(a-1)\ldots(a-r+2)(r-1) < 0. \qquad \square$$

Remark 3.8. The inequality $(r-1)d < n$ is not a necessary condition for the Cohen–Macaulayness of S_Δ. In fact, one guesses that S_Δ is Cohen–Macaulay if and only if $(r-1)d \le n$, which is indeed the case for $r = 2$ and $n \le 3$ (see Proposition 3.10 (4) below).

Remark 3.9. If I as above is a *radical* ideal generated by a regular sequence of r forms of the same degree, then S_Δ is a normal domain. This is a consequence of Corollary 2.11 plus the well-known fact that the Rees algebra of a radical normally torsion-free ideal is normal.

We illustrate the above results in the case $r = 2$. The diagonal subalgebras in this case include the coordinate rings of some rational surfaces in $\mathbb{P}^5$, obtained as the embedding of the blowup of $\mathbb{P}^2$ at the points of intersection of two plane curves of the same degree d by the linear system of forms of degree $d+1$ on $\mathbb{P}^2$ vanishing at these points (cf. [10]).

Proposition 3.10. *Let $I = (f_1, f_2) \subset k[\mathbf{X}] = k[X_1,\dots,X_n]$ be an ideal generated by a regular sequence of 2 forms of degree d. Let $S = \mathscr{R}(I)$ be the standard bigraded Rees algebra of I. Then*:

(1) *$S_\Delta \simeq k[\mathbf{U}]/\mathfrak{I}$ where $\mathbf{U} = (U_{ij})$ is a $n \times 2$ matrix of indeterminates and $\mathfrak{I}$ is the ideal minimally generated by the $\binom{n}{2}$ 2-minors of* **U** *and d forms of degree d.*

(2) *The Hilbert function of S_Δ is given by*

$$H_{S_\Delta}(u) = (u+1)\binom{u+n-1}{n-1} - u\binom{u-d+n-1}{n-1}.$$

Moreover, $\dim S_\Delta = n$ *and* $e(S_\Delta) = 1 + d(n-1)$. *Further, the Hilbert series of S_Δ is*

$$P_{S_\Delta}(z) = \frac{1 + nz + \cdots + nz^{d-1} + (n-d)z^d}{(1-z)^n}.$$

and, if $n = 3$, we have

$$H_{S_\Delta}(u) = (u+1)\binom{u+2}{2} - u\binom{u-d+1}{2},$$

(*compare with* [10, *Proposition* V.1]).

(3) *S_Δ is a Cohen–Macaulay ring if $d < n$ and not if $d > n$ (See also (4) below for the cases $n = 2, 3$, where S_Δ is Cohen–Macaulay for $d = n$).*

(4) *Let $n = 2$ (resp. $n = 3$). Then $k[\mathbf{U}]/\mathfrak{I}$ is the coordinate ring of a curve C (resp. a surface V) of degree $d + 1$ (resp. $2d + 1$) in $\mathbb{P}^3$ (resp. in $\mathbb{P}^5$) and the structure sheaves admit the following free resolutions as $\mathcal{O}_{\mathbb{P}}$-modules, respectively*:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-d-2)^{d-2} \to \mathcal{O}_{\mathbb{P}^3}(-d-1)^{2(d-1)} \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-d)^d$$
$$\to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_C \to 0.$$

and

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-d-3)^{d-3} \to \mathcal{O}_{\mathbb{P}^5}(-d-2)^{3(d-2)} \to \mathcal{O}_{\mathbb{P}^5}(-3)^2 \oplus \mathcal{O}_{\mathbb{P}^5}(-d-1)^{3(d-1)}$$
$$\to \mathcal{O}_{\mathbb{P}^5}(-2)^3 \oplus \mathcal{O}_{\mathbb{P}^5}(-d)^d \to \mathcal{O}_{\mathbb{P}^5} \to \mathcal{O}_V \to 0.$$

Proof. (1) This follows from the general description given before. The d forms of degree d can be explicitely described as follows: Consider the d elements $(f_1T_2 - f_2T_1)T_1^{a_1}T_2^{a_2}$, $a_1 + a_2 = d - 1$, and take a set of preimages of these under the map $U_{ij} \to X_iT_j$ (compare with [10, Theorem V.2]).

(2) This follows from Theorem 3.6.

(3) This follows from Theorem 3.7.

(4) By part (1), C is a divisor on a quadric W which is defined by the determinant of the matrix **U**. The resolution of $\mathcal{O}_W$ as an $\mathcal{O}_{\mathbb{P}^3}$-module being well-known, the mapping cone of the morphism of complexes given by the embedding $\mathcal{O}_W(-C) \to \mathcal{O}_W$ (see [21]), yields the above resolution of $\mathcal{O}_C$ as an $\mathcal{O}_{\mathbb{P}^3}$-module. By looking at the Hilbert function of $\mathcal{O}_C = S_\Delta$ or by a direct computation, one sees that the resolution is minimal.

For $n = 3$, one considers $k[\mathbf{U}]/\mathfrak{I}$ as the coordinate ring of a surface V of degree $2d+1$ in $\mathbb{P}^5$, which is a divisor on the rational normal scroll T defined by the ideal $I_2(\mathbf{U})$. It is known that

$$Pic(T) = \mathbb{Z}H \oplus \mathbb{Z}R,$$

where H is the hyperplane class and R is the ruling. Further the following equality holds in $Pic(T)$:

$$V = dH - (d-1)R.$$

From this, as above, one derives a minimal free resolution of $\mathcal{O}_V$ as an $\mathcal{O}_{\mathbf{P}^5}$-module as stated. In the case where I is the ideal of a set of points in $\mathbf{P}^2$ obtained as the complete intersection of two curves of degree d, this has been proved by Holay [18].

As a consequence of the above minimal resolutions, if $n = 2,3$, S_Δ is Cohen–Macaulay if and only if $d \leq n$ (cf. [10]). □

3.3. Determinantal ideals

This part concerns algebras with straightening law. For the relevant notation and definitions regarding this subject, we refer to [5, 6] (see also [4]). As usual, if R is a graded k-algebra with straightening law on a finite poset Π generating R, then one identifies the elements of Π with the corresponding elements of R indexed by them. In particular, an element of Π has a certain degree if the corresponding homogeneous element of R has that degree.

We first state a particular case of [20, Theorem 1.4] in the form that suits us best.

Proposition 3.11. *Let R be a graded k-algebra with straightening law on a finite poset Π. Let $\Omega \subset \Pi$ be a straightening closed poset ideal and let $\omega_1, \ldots, \omega_r$ be a linearization of the elements of Ω, all assumed to be of the same degree. If $I = \Omega R$ and $J \subset R[\mathbf{T}] = R[T_1, \ldots, T_r]$ denotes the presentation ideal of the Rees algebra $\mathcal{R}(I)$, then*

$$J^* = ((\Pi^{\omega_1}R)T_1, \ldots, (\Pi^{\omega_r}R)T_r) + (T_{j_1}T_{j_2} \mid \omega_{j_1}, \omega_{j_2} \in \Omega, \omega_{j_1} \not\sim \omega_{j_2})$$

where J^* is the ideal generated by the initial forms of the elements of J with respect to the lexicographic term order on $R[\mathbf{T}]$.

In the above statement, for an element $\pi \in \Pi$, $\Pi^\pi = \{\sigma \in \Pi \mid \sigma \not\geq \pi\}$ (the ideal *cogenerated* by π), while the symbol $\not\sim$ indicates incomparability relation.

Corollary 3.12. *With the notation of Proposition* 3.11, *if besides $R = k[\mathbf{X}]$, one has*

(i) $J^* = \bigcap_{j=1}^r Q_j$, *where* $Q_j = (\Pi^{\omega_j}R, T_k, T_{j_1}T_{j_2} \mid \omega_k \not\leq \omega_j, \omega_{j_1} < \omega_j, \omega_{j_2} < \omega_j, \omega_{j_1} \not\sim \omega_{j_2})$.

(ii) $k[\mathbf{X},\mathbf{T}]/Q_j \simeq k[\mathbf{X}]/\Pi^{\omega_j}R \otimes_k k[\Delta_j]$, *where Δ_j denotes the order simplicial complex of the subposet $\Pi_{\omega_j} \cup \{\omega_j\} = \{\sigma \in \Pi \mid \sigma \leq \omega_j\}$.*

Proof. The proof is the same as in [20, Proof of Theorem 2.2], if one observes that here $(0 : \omega_1) = (0)$. □

Here is the main result of this portion which gives a template for straightening closed ideals.

Theorem 3.13. *Let $R = k[\mathbf{X}]$ (with the standard gradation) admit a structure of monotonely graded algebra with straightening law on an upper semimodular semilattice Π and let $\Omega \subset \Pi$ be a straightening closed ideal such that* $\operatorname{rank} \Pi \setminus \Omega \geq 2$ *and all elements of Ω have the same degree. Then the diagonal subalgebra of the bigraded Rees algebra $\mathscr{R}(\Omega R)$ is Cohen–Macaulay.*

Proof. We invoke the same strategy as the one in Subsection 2.3 (cf. also the proof of Theorem 3.7).

Let $\omega_1, \ldots, \omega_r$ be a linearization of the elements of Ω and choose presentation variables $\mathbf{T}$ as in Proposition 3.11. Using the explicit decomposition of Corollary 3.12 and the notation there, we first claim that $A_\Delta/(Q_j)_\Delta$ is Cohen–Macaulay for every $1 \leq j \leq r$, where $A = k[\mathbf{X}, \mathbf{T}]$. Indeed, by part (ii) of that corollary, we are to show that the Segre product $R/\Pi^{\omega_j}R \underline{\otimes}_k k[\Delta_j]$ is Cohen–Macaulay.

Let us first argue that $R/\Pi^{\omega_j}R$ and $k[\Delta_j]$ are Cohen–Macaulay. For the first, one observes that it is a graded algebra with straightening law on the poset $\Pi \setminus \Pi^{\omega_j}$ since Π^{ω_j} is an ideal of Π. Moreover, the assumptions certainly imply that Π is locally upper semimodular. Since $\Pi \setminus \Pi^{\omega_j}$ has a unique minimal element, it follows that it is locally upper semimodular too [6, (5.13) (a)]. Therefore, $R/\Pi^{\omega_j}R$ is Cohen–Macaulay [6, (5.14)].

As for $k[\Delta_j]$, it suffices by [2] to show that the poset $\Pi_{\omega_j} \cup \{\omega_j\}$ is locally upper semimodular. Thus, let $\pi_1, \pi_2 \in \Pi_{\omega_j} \cup \{\omega_j\}$ be covers of $\sigma \in \Pi_{\omega_j} \cup \{\omega_j\}$ and let $\tau \in \Pi_{\omega_j} \cup \{\omega_j\}$ be such that $\tau > \pi_1$, $\tau > \pi_2$. Since Π is assumed to be an upper semimodular semi-lattice, $\pi_1 \sqcup \pi_2$ is a common cover of π_1 and π_2 and, clearly, $\pi_1 \sqcup \pi_2 \leq \tau$, hence $\pi_1 \sqcup \pi_2 \in \Pi_{\omega_j} \cup \{\omega_j\}$, as required.

According to [23, Theorem], the Segre product $R/\Pi^{\omega_j}R \underline{\otimes}_k k[\Delta_j]$ is Cohen–Macaulay if $\rho(k[\Delta_j]) \leq \iota(R/\Pi^{\omega_j}R)$ and $\rho(R/\Pi^{\omega_j}R) \leq \iota((k[\Delta_j])$, where $\rho(G)$ and $\iota(G)$ denote, respectively, the Hilbert function regularity index and the initial degree of the standard graded k-algebra G. It is well known that the face ring of a simplicial complex has regularity index 0, hence the first inequality is trivially verified. As to the second inequality, by [4, Theorem 1.1 and Corollary 1.3 (a)] we have that the a-invariant of $R/\Pi^{\omega_j}R$ is negative. Therefore, by the well-known relation between the two invariants, we deduce that $\rho(R/\Pi^{\omega_j}R) \leq 0\,(= \iota(k[\Delta_j]))$.

We are thus left with the low dimensional cases of $R/\Pi^{\omega_j}R$ or $\dim k[\Delta_j]$. The latter could only happen if the elements $\omega_1, \ldots, \omega_j$ were all mutually incomparable, but this is impossible because of the straightening law and the fact that R has no proper zero divisors. As to the former, one has

$$\dim R/\Pi^{\omega_j}R = \operatorname{rank} \Pi - \operatorname{rank} \Pi^{\omega_j} \geq \operatorname{rank} \Pi - \operatorname{rank} \Omega \geq 2,$$

by assumption, so it can not take place either. This completes the proof that the Segre product of $R/\Pi^{\omega_j}R$ and $k[\Delta_j]$ is Cohen–Macaulay.

We next deal with the Cohen–Macaulayness of $Q_1 \cap \ldots \cap Q_j + Q_{j+1}$. A moment's reflexion yields that this ideal is generated by $\Pi^{\omega j+1}$, T_k, $T_{j_1}T_{j_2}$ with $\omega_k \not\leq \omega_{j+1}$, $\omega_{j_1} \not\sim \omega_{j_2}$, $\omega_{j_1} < \omega_{j+1}$, $\omega_{j_2} < \omega_{j+1}$. It follows that $A_\Delta/((Q_1)_\Delta \cap \ldots \cap (Q_j)_\Delta + (Q_{j+1})_\Delta)$ is isomorphic to the Segre product of $R/\Pi^{\omega_{j+1}}R$ and the face ring of the order complex of $\Pi_{\omega_{j+1}} \cup \{\omega_{j+1}\}$. Therefore, we are back to the same situation as above, hence this ring is Cohen–Macaulay.

Thus, $A_\Delta/(J^*)_\Delta$ is Cohen–Macaulay by Lemma 2.8. □

Corollary 3.14. *Let* $\mathbf{X} = (X_{ij})$ *denote a matrix of indeterminants over the field* k, *let* $R = k[\mathbf{X}]$ *and let* $I \subset R$ *denote the ideal generated by the maximal minors of* $\mathbf{X}$. *Then the diagonal subalgebra* $k[(\mathbf{X})I_d]$ *of the standard bigraded Rees algebra* $\mathcal{R}(I)$ *is a Cohen–Macaulay normal domain.*

Proof. Let Π denote the poset of all minors of $\mathbf{X}$ and let $\Omega \subset \Pi$ denote the ideal of all maximal minors of $\mathbf{X}$. Say, $\mathbf{X}$ is a $d \times c$ matrix with $d \leq c$. If $d = 1$ then $\mathcal{R}(I)_\Delta$ is isomorphic to the k-subalgebra of $k[\mathbf{X}]$ generated by the monomials of degree 2 in the variables $\mathbf{X}$. This is well known to be Cohen–Macaulay (e.g., because it is normal). Thus, we may assume that $d \geq 2$. In this case, it is easy to check that $\operatorname{rank} \Pi \setminus \Omega \geq 2$. Furthermore, it is well known that $R = k[\mathbf{X}]$ has a structure of algebra with straightening law on Π satisfying the hypotheses of Theorem 3.13 (Π is actually a distributive lattice [6]) and Ω is a straightening closed ideal for this structure (see [5, 6]). This proves the Cohen–Macaulayness of the diagonal subalgebra of $\mathcal{R}(I)$. Its normality follows, by Corollary 2.11, from the fact that $\mathcal{R}(I)$ is normal [5]. □

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